3.17 Imagine assembling the electron cloud layer by layer, from the center outwards. The energy to add the layer of charge that lies between \( r \) and \( r + dr \) is \( \varphi'(r)\rho(r)4\pi r^2 dr \), where \( \varphi'(r) \) is the potential at \( r \) due to all the charge inside the new layer. Since all the charge inside this layer can be treated as being at the origin,

\[
dE = 4\pi r^2 dr \times \left[ \frac{|e|}{\pi a^2 r} e^{-2r/a} \right] \times \left[ \frac{1}{4\pi \varepsilon_0} Q(< r) \right]
\]  

where

\[
Q(< R) = |e| - \int_0^R \frac{|e|}{\pi a^2 r} e^{-2r/a} 4\pi r^2 dr = \int_R^\infty \frac{e}{\pi a^2 r} e^{-2r/a} 4\pi r^2 dr = |e| e^{-2R/a} (1 + 2R/a).
\]

Therefore

\[
dE = -\frac{e^2}{\pi \varepsilon_0 a^2} e^{-2r/a} e^{-2r/a} (1 + 2r/a) dr.
\]

Integrating this expression from \( r = 0 \) to \( r = \infty \), the total energy required to assemble the charge distribution is

\[
\frac{e^2}{8\pi \varepsilon_0 a^2} [a/4 + a/8] = -\frac{3e^2}{8\pi \varepsilon_0 a}.
\]

The negative of this is, therefore, the energy required to disassemble the charge distribution.

3.20 If we parametrize the line segment as instructed, the distance from the observation point to the point \( s' = 0 \) is \( |(b \times a)/a| \), the projection of \( b \) perpendicular to \( a \). And at the two limits of the integral, we have \( s' = c \cdot a/a \). Therefore the potential is

\[
\varphi(r) = \frac{\lambda}{4\pi \varepsilon_0} \int \frac{ds'}{\sqrt{s'^2 + b \times a^2/a^2}} = \frac{\lambda}{4\pi \varepsilon_0} \ln[s' + \sqrt{s'^2 + |b \times a|^2/a^2}]_{s'=b/a}^{s'=c/a}
\]

from which the desired result follows. It is possible to replace the \( |b \times a|/a \) in the answer with \( |c \times a|/a \), because \( b - c = a \) and \( a \times a = 0 \).

4.2 The charge density is effectively \( (\pi/2) \exp(\kappa z) \) for \( z < 0 \) and \( -(\pi/2) \exp(-\kappa z) \) for \( z > 0 \). The dipole moment per unit area is, therefore

\[
p_z/A = \int_{-\infty}^{\infty} z \frac{\pi}{2} e^{\kappa z} dz - \int_{0}^{\infty} z \frac{\pi}{2} e^{-\kappa z} dz = -\pi/\kappa^2.
\]

The electric field satisfies \( dE_z/dz = -\rho(z)/\varepsilon_0 \). By inspection, one can see that \( E_z = (\pi/2\kappa \varepsilon_0) \exp(-\kappa z) \) for \( z > 0 \), and is an even function of \( z \). The potential satisfies \( d\varphi/dz = -E_z \). By symmetry, the potential is zero at the origin.
Therefore
\[
\varphi(z) = -\frac{\pi}{2\kappa e_0} \int_0^z e^{-\kappa z'} dz' = \frac{\pi}{2\kappa^2 e_0} (e^{-\kappa z} - 1) \tag{7}
\]
for \(z > 0\), and \(\varphi(-z) = -\varphi(z)\). The potential difference \(\varphi(\infty) - \varphi(-\infty)\) is equal to \(\pi/(\kappa^2e)\). This is because the charge distribution looks like an infinite dipole layer from far away, for which the potential difference between the two sides is \(p_z/(\kappa e_0)\). (Think of the infinite parallel plate capacitor.)

The electrostatic energy per unit area is
\[
\frac{1}{2} \int_{-\infty}^{\infty} \rho(z)\varphi(z) dz = \int_{0}^{\infty} \rho(z)\varphi(z) dz = -\frac{\pi^2}{4\kappa^2 e_0} \left[ 1/(2\kappa) - 1/\kappa \right] = \frac{\pi^2}{8\kappa^3 e_0}. \tag{8}
\]

4.14 a) The net charge is zero. Breaking the four charges into two pairs, we see that there are two dipoles, equal and opposite in orientation but slightly displaced with respect to each other. Therefore the dipole moment is zero. For the quadrupole moment, the only non-zero components of \(Q_{ij}\) can be those for which \(i, j \neq 3\). Since all the charges have the same value of \(x^2\) and \(y^2\), the only non-zero component is \(Q_{12} = 2qa^2\).

b) The monopole moment is \(2M\). Since the center of charge is at the origin, the dipole moment is zero. For the quadrupole moment, the only non-zero component is \(Q_{33} = (\int \lambda z^2 dz)/2 = \lambda l^3/3\).

c) The monopole moment is \(2\pi R\lambda\), and the dipole moment is zero because the center of charge is at the origin. For the quadrupole moment, since the charge distribution is symmetric under \(x \to -x\) and has \(z = 0\), the non-zero components are \(Q_{11}\) and \(Q_{22}\). By symmetry, these two are equal. Since \(Q_{11} + Q_{22}\) is trivially \(2\pi R\lambda \times R^2\), each of them is \(\pi R^3\lambda\).

5. By inspection, the charge density can be written in terms of the spherical harmonics as
\[
\rho(r') = e^{-r'} \left[ \frac{4}{5} \sqrt{\frac{\pi}{3}} Y_{20} + \frac{4}{3} \sqrt{\pi} Y_{00} \right]. \tag{9}
\]
The potential at a point \(r\) from the charge for which \(r' < r\) is
\[
\varphi_<(r) = \frac{1}{\epsilon_0 r} Y_{00}(\theta, \phi) \frac{4}{5} \sqrt{\pi} \int_0^r r'^2 e^{-r'} dr' + \frac{1}{5\epsilon_0 r^3} \frac{4}{3} \sqrt{\pi} Y_{20}(\theta, \phi) \int_0^r r'^4 e^{-r'} dr'. \tag{10}
\]
The potential from the charge for which \(r' > r\) is
\[
\varphi_>(r) = \frac{1}{\epsilon_0 r} Y_{00}(\theta, \phi) \frac{4}{5} \sqrt{\pi} \int_r^{\infty} r'^2 e^{-r'} dr' + \frac{r^2}{5\epsilon_0 r^3} \frac{4}{3} \sqrt{\pi} Y_{20}(\theta, \phi) \int_r^{\infty} \frac{1}{r'} e^{-r'} dr'. \tag{11}
\]
Adding these up, and evaluating the integrals, the spherically symmetric part of the potential is
\[
\varphi_0(r) = \frac{2}{3\epsilon_0} \left[ \frac{2 - (r^2 + 2r + 2)e^{-r}}{r} + (r + 1)e^{-r} \right] = \frac{2}{3\epsilon_0} \frac{2 - (r + 2)e^{-r}}{r}. \tag{12}
\]
For the quadrupole term, we use the asymptotic series from repeated integration by parts
\[
\int_0^\infty r e^{-r} dr' = e^{-r}(1/r - 1/r^2 + 2/r^3 - 6/r^4 \ldots)
\]  
(13)
to obtain
\[
\varphi_2(r) = \frac{1}{15\varepsilon_0}(3 \cos^2 \theta - 1)(24/r^3 - e^{-r}(24/r^3 + 24/r^2 + 12/r + 4 + r)) + e^{-r}(1/r - 1/r^2 + 2/r^3 \ldots)].
\]  
(14)
As \( r \to \infty \), the leading contributions to \( \varphi_0 \) and \( \varphi_2 \) are \( \propto 1/r \) and \( \propto 1/r^3 \) respectively.

6. Expressing the potential due to \( \rho' \) in an interior multipole expansion,
\[
\varphi(r) = \frac{1}{\epsilon_0} \sum_{lm} \frac{1}{2l+1} r^l Y_{lm}(\theta, \phi) \int Y_{lm}^{*}(\theta', \phi') \rho(r') dr'
\]  
(15)
and therefore the interaction energy is
\[
E = \frac{1}{\epsilon_0} \sum_{lm} \frac{1}{2l+1} \int r^l Y_{lm}(\theta, \phi) \rho(r) dr \int \frac{Y_{lm}^{*}(\theta', \phi')}{{r'}^{l+1}} \rho(r') dr'.
\]  
(16)
If the charge density \( \rho(r) \) is spherically symmetrical, the only term in the first integral that is non-zero is the one with \( l = m = 0 \). In that case, recalling that \( Y_{00}(\theta, \phi) = 1/\sqrt{4\pi} \), we have
\[
E = \int \rho(r) dr \times \frac{1}{4\pi\varepsilon_0} \int \frac{1}{r'} \rho(r') dr'.
\]  
(17)
The second part of this expression is clearly the electrostatic potential due to \( \rho'(r) \) at the origin.