Physics 212: Solutions 2

1. The solution to \( \frac{d^4y}{dx^4} = \delta(x - x') \) is cubic polynomial in \( x \) for \( x < x' \) and a different cubic polynomial for \( x > x' \). Applying the boundary conditions, \( y(x < x') = Ax^2 + Bx^3 \), while \( y(x > x') = C + Dx \). From the continuity of the function, its first and second derivatives at \( x = x' \), we have

\[
\begin{align*}
Ax'^2 + Bx'^3 &= C + Dx' \\
2Ax' + 3Bx'^2 &= D \\
2A + 6Bx' &= 0.
\end{align*}
\]

The discontinuity in the third derivative of the function implies that

\[
0 - 6B = 1.
\]

Therefore \( B = -1/6 \), \( A = x'/2 \), \( D = x'^2/2 \) and \( C = -x'^3/6 \).

2. i) If the charges are \( \pm q \) located at \( z = \pm \delta/2 \), i.e. the dipole moment is \( p = q\delta \) in the positive \( z \) direction, the image charges are \( \mp q(2R/\delta) \) located at \( z = \pm 2R^2/\delta \). Because these charges are so far away, the field due to them everywhere inside the sphere is the same as the field at the center:

\[
\frac{2}{4\pi \epsilon_0} \frac{q(2R/\delta)}{(2R^2/\delta)^2}
\]

in the positive \( z \) direction. Simplifying, this is equal to \( 1/(4\pi \epsilon_0)(q\delta)/R^3 = p/(4\pi \epsilon_0 R^3) \) in the positive \( z \) direction.

ii) The electrostatic potential on the surface of the sphere due to the dipole itself is

\[
\varphi_0(R, \theta) = \frac{pR \cos \theta}{4\pi \epsilon_0 R^3}.
\]

To this, we must add a solution of Laplace’s equation inside the sphere that satisfies \( \varphi_1 = -\varphi_0 \) on the surface of the sphere. Expanding in spherical harmonics, because of azimuthal symmetry,

\[
\varphi_1(r, \theta) = \sum_l C_l r^l Y_{l0}(\theta).
\]

Since \( Y_{l0} \propto \cos \theta \), we conclude that \( \varphi_1(r, \theta) = K_1 r \cos \theta \). Applying the condition \( \varphi_1 = -\varphi_0 \) at \( r = R \), we have

\[
\varphi_1(r, \theta) = -\frac{pr \cos \theta}{4\pi \epsilon_0 R^3}.
\]

Replacing \( r \cos \theta \) with \( z \), the electric field has only a \( z \) component, equal to \( p/4\pi \epsilon_0 R^3 \). This is the same as we got with image charges.

8.12 a) Let \( \Phi(\rho, \phi) \) be a solution to Laplace’s equation inside the cylinder. This means that

\[
\rho \partial_\rho (\rho \partial_\rho \Phi) + \partial_\phi^2 \Phi = 0 \quad \rho < R.
\]
In terms of \( u = 1/\rho \), we have \( \rho \partial_\rho = \partial_{\ln \rho} \). Therefore

\[
u \partial_u (u \partial_u \Phi) + \partial_\rho^2 \Phi = 0 \quad u > R.
\] (8)

Thus we obtain a solution to Laplace’s equation outside the cylinder.

b) Let \( \Phi_0 \) be the potential due to a line charge located at \( \rho > R \), in free space (i.e. no conducting cylinder). If we restrict ourselves to \( \Phi_0 \) inside the cylinder, it clearly satisfies Laplace’s equation. Turning this restricted solution inside out, as in part a), \( \Psi_0(\rho > R, \phi) \) is a solution to Laplace’s equation outside the cylinder. And clearly \( \Psi_0(R, \phi) = \Phi_0(R, \phi) \). Therefore \( \Phi_0(\rho > R, \phi) - \Psi_0(\rho > R, \phi) \) is a solution to Poisson’s equation for the line charge, and is zero on the cylinder. Therefore this is the solution to the Dirichlet boundary value problem outside the cylinder with the cylinder being grounded.

If the line charge is inside the cylinder, one can do the same thing in reverse: consider the solution to Poisson’s equation for the line charge in free space, observe that the part of this solution that is restricted outside the cylinder solves Laplace’s equation, turn this part of the solution inside out to be a solution of Laplace’s equation inside the cylinder, and subtract from the original solution inside the cylinder to obtain a solution to Poisson’s equation inside the cylinder that is zero on the cylinder.

c) This was assigned by mistake, since it involves dielectrics.

8.19 a) This problem was worked out in class, but with one corner of the box at the origin instead of the center of the box.

b) Let us consider the potential due to a point charge \(-Q\) at the center of the box, with the box grounded. This is the Coulomb potential due to the charge \(-Q\) and the Coulomb potential due to the charge on the surface of the box. Outside the box, the potential is zero. This means that, outside the box, the potential due to the surface charge exactly cancels the potential due to the charge \(-Q\). This means that the potential (and field) outside the box due to the surface charge is equal to the potential (and field) due to a charge \(+Q\) at the center of the box.

How much is this surface charge? The Green’s function for the box \( 0 < x, y, z < 2a \) is

\[
G(r, r') = \frac{1}{a^2 \epsilon_0} \sum_m \sum_n \sin \left( \frac{n \pi x}{2a} \right) \sin \left( \frac{n \pi x'}{2a} \right) \sin \left( \frac{m \pi y}{2a} \right) \sin \left( \frac{m \pi y'}{2a} \right) \frac{\sinh(\kappa_{nm} z) \sinh(\kappa_{nm} (2a - z))}{\kappa_{nm} \sinh(2 \kappa_{nm} a)}
\] (9)

where

\[
\kappa_{nm}^2 = (n^2 + m^2) \pi^2 / (4a^2).
\] (10)

The potential due to a point charge \(-Q\) at \((a, a, a)\) is therefore

\[
\varphi(r) = -\frac{Q}{a^2 \epsilon_0} \sum_{m,n=1,3,...} \sin \left( \frac{n \pi x}{2a} \right) \sin \left( \frac{m \pi y}{2a} \right) (-1)^{(m-1)/2} (-1)^{(n-1)/2} \frac{\sinh(\kappa_{nm} a) \sinh(\kappa_{nm} (a - z - a))}{\kappa_{nm} \sinh(2 \kappa_{nm} a)}
\] (11)
The surface charge on the $z = 2a$ surface is then obtained as $\epsilon_0 \partial_z \varphi$ on that surface, i.e.

$$\sigma(x, y, 2a) = \frac{Q}{a^2} \sum_{m, n=1, 3, \ldots} \sin\left(\frac{n\pi x}{2a}\right) \sin\left(\frac{m\pi y}{2a}\right) (-1)^{(m-1)/2} (-1)^{(n-1)/2} \frac{\sinh(\kappa a)}{\sinh(2\kappa a)}. \quad (12)$$

In particular,

$$\sigma(a, a, 2a) = \frac{Q}{2a^2} \sum_{m, n=1, 3, \ldots} \frac{1}{\cosh(\kappa nm a)} = \frac{Q}{2a^2} \sum_{m, n=1, 3, \ldots} \frac{1}{\cosh(\pi/2)\sqrt{m^2 + a^2}} \approx 0.123 \frac{Q}{a^2}. \quad (13)$$

8.20 a) This was worked out in class.

b) To see that this is what we would get with the method of image charges, note that the first term is the Coulomb potential at $r'$ due to a unit point charge at $r$. For the Coulomb potential at $r'$ due to the image charge, note that $r_I = R^2/r$ is always greater than $r'$ because $r' < R$ is inside the sphere. Therefore the potential due to the image charge is

$$\frac{q_I}{4\pi\epsilon_0} \sum_{l} \frac{r'^l}{r'^l+1} P_l = -\frac{R}{4\pi\epsilon_0} \sum_{l} \frac{r'^l(r'^l+1)}{R^{2l+2}} = -\frac{1}{4\pi\epsilon_0} \sum_{l} \frac{r'^l(r'^l)}{R^{2l+1}}. \quad (14)$$

This is identical to the second term in the Green’s function.