Physics 212: Fall 2016
Homework 3 Solutions

1. The sphere cannot be neutral because there is an attractive force between a point charge and any isolated neutral object. According to image theory, a grounded sphere acquires a charge \( q' = -qR/s \) where \( s \) is the distance between \( q \) and the center of the sphere. The image itself lies a distance \( d = R^2/s \) from the center. Now, suppose we add a second image charge \( q'' = Q - q' \) at the center of the sphere. By Gauss’ law, the total charge on the sphere is now \( Q \) and the sphere boundary is still an equipotential. This is the situation we want.

We have \( s = 2R \), so \( q' = -q/2 \), \( s = R/2 \), and \( q'' = Q + q/2 \). We need to choose \( Q \) so that the force between \( q \) and the two images is zero:

\[
F = \frac{q^2}{4\pi\epsilon_0} \left\{ \frac{q'}{(s-d)^2} + \frac{q''}{s^2} \right\} = \frac{q}{4\pi\epsilon_0} \left\{ \frac{-q/2}{(2R-R/2)^2} + \frac{Q + q/2}{(2R)^2} \right\} = 0.
\]

(1)

This gives \( Q = 7q/18 \). When we move \( q \) so that \( s = 3R \), the force formula on the left is still correct with \( q' = -q/3 \), \( d = R/3 \), and \( q'' = Q + q/3 \). Therefore, the force is

\[
F = \frac{q^2}{4\pi\epsilon_0} \left\{ \frac{-1/3}{(3R-R/2)^2} + \frac{7/18 + 1/3}{(3R)^2} \right\} = \frac{\lambda}{2\pi\epsilon_0} \frac{R^2}{173 \cdot 5184}.
\]

(2)

2. The cylinder is an equipotential if the line charge \( \lambda \) at a distance \( b = 2R \) has an image line charge of strength \( -\lambda \) placed at a distance \( R^2/b = R/2 \) from the center. In cylindrical coordinates, the potential is then

\[
\varphi(\rho, \phi) = \frac{\lambda}{4\pi\epsilon_0} \ln \frac{\rho^2 + R^2/4 - \rho R \cos \phi}{\rho^2 + 4R^2 - 4\rho R \cos \phi},
\]

(3)

and one can verify that the potential is independent of \( \phi \) when \( \rho = R \). The surface charge density is then

\[
\sigma = -\epsilon_0 \partial_\phi \varphi |_{\rho=R} = -\frac{\lambda}{2\pi R} \left( \frac{3}{5 - 4 \cos \phi} \right).
\]

(4)

The total force per unit length on the surface is

\[
F = \frac{1}{2\pi\epsilon_0} \int \rho d\phi \sigma^2(\phi) \hat{\rho} = \frac{R}{2\pi\epsilon_0} \int d\phi \sigma^2(\phi) \cos \phi = \frac{\lambda^2}{3\pi\epsilon_0 R} \hat{x}.
\]

(5)

The force can also be computed as the equal and opposite of the force that acts on the line charge. Since the distance between the line charge and its image is \( 3R/2 \), the force per unit length is

\[
\frac{\lambda^2}{2\pi\epsilon_0 (3R/2)}
\]

(6)
in the negative \( x \) direction. Therefore the force on the cylinder is the same, but in the positive \( x \) direction.
3. Actually, it is easier to disregard the hint and use Coulomb’s law for the potential:

\[ \varphi(x, y, z > 0) = \frac{A}{4\pi\epsilon_0} \int \frac{1}{[z^2 + (x - x')^2 + (y - y')^2]^{1/2}} \left[ \frac{1}{R^2 - (x'^2 + y'^2)} \right]^{1/2} dx' dy'. \]  

(7)

At the origin, the integral is easy to perform in cylindrical coordinates:

\[ \varphi(0, 0, 0) = \frac{A}{2\epsilon_0} \int \frac{1}{\rho} \frac{1}{[R^2 - \rho^2]^{1/2}} \rho d\rho = \frac{\pi A}{4\epsilon_0}. \]  

(8)

The total charge of the disk is

\[ 2\pi A \int \frac{1}{[R^2 - \rho^2]^{1/2}} \rho d\rho = 2\pi AR. \]  

(9)

Therefore the potential on the surface of the disk is \( Q/(8\epsilon_0 R) \), and the capacitance is \( 8\epsilon_0 R \).

4. The electrostatic potential between the spherical shells is of the form \( \varphi(r, \theta, \phi) = \sum_{lm} Y_{lm}(\theta, \phi)(a_{lm} r^l + b_{lm} r^{l+1}) \). Since the radial fields on the boundary surfaces are proportional to \( Y_{00} \) and \( Y_{11} \), we can simplify this to

\[ \varphi(r, \theta) = [A_{00} + B_{00}/r] + [A_{10} r + B_{10}/r^2] \cos \theta. \]  

(10)

The radial component of the field is then

\[-\partial_r \varphi = B_{00}/r^2 + [2B_{10}/r^3 - A_{10}] \cos \theta. \]  

(11)

Applying the boundary conditions, we see that \( B_{00} = 0, A_{00} \) is not determined, \( 2B_{10}/r_0^3 = A_{10} \) and \( 2B_{10}/R^3 - A_{10} = -E_0 \). Therefore

\[ \varphi(r, \theta) = \frac{E_0 r}{1 - (r_0/R)^3} [1 + \frac{1}{2}(r_0/r)^3]. \]  

(12)

Actually, we could have started from the beginning with the fact that the electric field is only proportional to \( Y_{10} \) on both surfaces (zero is proportional to everything), but keeping \( A_{10} \) and \( B_{10} \) taught us something: unlike Poisson’s equation problems with Dirichlet boundary conditions, the solution with Neumann boundary conditions (when the normal component of the field is specified at the boundaries) is only obtained up to a proportionality constant, and the boundary conditions are not arbitrary: they must satisfy Gauss’ law. Thus in this case, if the radial field on the inner surface had been a non-zero constant, \( B_{10} \) would have to satisfy two mutually inconsistent equations. This is a result of Gauss’ law: if there is a net electric flux emerging from the inner surface, and there is no charge in the cavity, it cannot disappear at the outer surface.
5. Let the wires be at \( x = \pm R/2 \). Then the potential at any point is

\[
\varphi(x, y) = -\frac{\lambda}{4\pi\epsilon_0} \ln \frac{(x - R/2)^2 + y^2}{(x + R/2)^2 + y^2}.
\]

The equipotential surface where \( \varphi = \lambda A/(4\pi\epsilon_0) \) satisfies the equation

\[
(x - R/2)^2 + y^2 = e^{-A}[(x + R/2)^2 + y^2].
\]

This is equivalent to

\[
(1 - e^{-A})[x^2 + y^2 + R^2/4] = (1 + e^{-A})Rx.
\]

This is clearly the equation of a circle, whose center is on the \( x \)-axis. By dividing throughout by \( 1 - e^{-A} \) and completing the square, one can see that the center of the circle is at

\[
x = \frac{R}{2} \frac{1 + e^{-A}}{1 - e^{-A}} = \frac{R}{2} \coth(A/2)
\]

and the radius of the circle satisfies

\[
\rho^2 = \frac{R^2}{4} \left[ \coth^2(A/2) - 1 \right] = \frac{R^2}{4 \sinh^2(A/2)}.
\]

By judiciously choosing the separation between the two wires, we can ensure that two circles (cylinders) of just the right sizes and the right separation are equipotentials. Then, so far as the region between the circles is concerned, if we replace the wires with the equipotential circles, the solution to Laplace’s equation is unchanged. If the two circles correspond to \( A = A_1 \) and \( A = -A_2 \), then the conditions to be satisfied are

\[
d = \frac{R}{2} \left[ \coth A_1 + \coth A_2 \right]
\]

\[
a^2 = \frac{R^2}{4 \sinh^2(A_1/2)}
\]

\[
b^2 = \frac{R^2}{4 \sinh^2(A_2/2)}
\]

After this, it is straightforward algebraic manipulation to verify that

\[
d^2 - a^2 - b^2 = \cosh(A_1/2 + A_2/2).
\]

The potential difference between the two equipotential surfaces is \( (\lambda/4\pi\epsilon_0)(A_1 + A_2) \). Therefore the capacitance per unit length is \( 4\pi\epsilon_0/(A_1 + A_2) \) which is equal to

\[
\frac{2\pi\epsilon_0}{\cosh^{-1}(d^2 - a^2 - b^2)/(2ab)}.
\]