Physics 212 Fall 2016
Homework Solutions 4

5.7 The potential on the outermost shell can be obtained by pretending that all the charge is at the origin:

$$\varphi_c = \frac{e_a + e_b + e_c}{4\pi\varepsilon_0 c}. \quad (1)$$

Anywhere inside the outermost sphere, the potential due to that sphere is $e_c/(4\pi\varepsilon_0 c)$, i.e. the same as on the sphere. Similarly, anywhere inside the middle sphere, the potential due to that sphere is $e_b/(4\pi\varepsilon_0 b)$. Therefore the potential on the innermost sphere is

$$\varphi_a = \frac{1}{4\pi\varepsilon_0 c} \left[ \frac{e_a}{a} + \frac{e_b}{b} + \frac{e_c}{c} \right]. \quad (2)$$

If this is to be equal to zero when the inner sphere is grounded, and the charge on the inner sphere is then $e'_a$, we have

$$e'_a = -a(e_b/b + e_c/c). \quad (3)$$

The change in the potential of the outer sphere is then

$$\Delta \varphi_c = \frac{1}{4\pi\varepsilon_0 c} \left[ -a(e_b/b + e_c/c) - e_a \right] = -\frac{a}{4\pi\varepsilon_0 c} [e_a/a + e_b/b + e_c/c]. \quad (4)$$

5.23 a) Let us use standard spherical polar coordinates, and the sphere be cut along the equator. By symmetry, the net force on the upper hemisphere is in the $z$ direction. On any small patch of the hemisphere, the force exerted is equal to $\sigma^2/(2\varepsilon_0)$ in the outward radial direction. Therefore the total $z$ force on the northern hemisphere is

$$\frac{\sigma^2 \pi}{2\varepsilon_0} \int_0^{\pi/2} \cos \theta \sin \theta d\theta = \frac{\sigma^2 \pi r^2}{2\varepsilon_0}. \quad (5)$$

Using the fact that $V = Q/(4\pi\varepsilon_0 r) = \sigma r/\varepsilon_0$, the force is equal to $V^2 \pi \varepsilon_0 / 2$. b) In this case, let the outer sphere be negatively charged and the inner sphere be positively charged. The electric field is confined to the region between the spheres. The inner hemisphere feels a $z$-force as in part a), equal to

$$\frac{\sigma_a^2 \pi a^2}{2\varepsilon_0}. \quad (6)$$

For the outer sphere, the field immediately outside a small patch is zero and the field immediately inside is $-\sigma_b/(2\varepsilon_0)$ pointing radially outwards. Therefore the
The force on this patch is equal to $\frac{\sigma^2 \pi b^2}{2\epsilon_0}$ pointing radially inwards. By a process similar to part a) of this problem, the net force on this hemisphere is

$$\frac{\sigma^2 \pi b^2}{2\epsilon_0}$$

pointing in the negative $z$ direction. Recalling that $4\pi \sigma_a a^2 = -4\pi \sigma_b b^2 = Q$, the total force on the pair of hemispheres is

$$\frac{Q^2}{32\pi \epsilon_0 a^2} - \frac{Q^2}{32\pi \epsilon_0 b^2}$$

pointing in the $z$ direction.

3. If a matrix $M$ is changed by a small amount, the change in its inverse satisfies

$$[M + \delta M] \cdot [M^{-1} + \delta M^{-1}] = I.$$  

(9)

Expanding, and throwing away the term that is a product of infinitesimals, we have

$$M \delta M^{-1} + \delta M M^{-1} = 0.$$  

(10)

Multiplying throughout with $M^{-1}$ from the left, we obtain the desired result.

Let us imagine that the conductor is first moved with the charges on the conductors held constant, i.e. with the voltage sources that maintain the potentials on the conductors disconnected. Then the mechanical work extracted is clearly equal to the reduction in the internal energy, i.e.

$$W = -\frac{1}{2} Q^T \cdot \delta C^{-1} \cdot Q$$

(11)

where the $Q^T$ consists of all the $Q_i$’s arranged as a row vector, and the $Q$ consists of all the $Q_i$’s arranged as a column vector. Applying the identity we just derived,

$$W = \frac{1}{2} Q^T \cdot C^{-1} \cdot \delta C \cdot C^{-1} \cdot Q.$$  

(12)

Since $Q = C \cdot \varphi$, we have $\varphi = C^{-1} \cdot Q$, and $\varphi^T = Q^T \cdot C^{-1}$ because $C^{-1}$ is a symmetric matrix. Therefore

$$W = \frac{1}{2} \varphi^T \cdot \delta C \cdot \varphi$$  

(13)

thus proving the desired result.

4. If there is a charge $Q_1$ on the inner sphere and $Q_2$ on the outer sphere, then the total energy of the system is

$$U = \frac{1}{2} [Q_1 \varphi_1 + Q_2 \varphi_2] = \frac{1}{8\pi \epsilon_0} \left[ \frac{Q_1^2}{a} + \frac{Q_2^2}{b} + \frac{2Q_1 Q_2}{b} \right].$$  

(14)
Comparing to the definition that \( U = \frac{1}{2} \sum Q_i Q_j C_{ij} \) we have \( C_{11}^{-1} = 1/(8\pi\epsilon_0 a) \) and \( C_{21}^{-1} = C_{22}^{-1} = 1/(8\pi\epsilon_0 b) \). To find the capacitance of the pair of spheres, assume that \( Q_1 = -Q_2 = Q \). Then

\[
U = \frac{1}{2} Q^2 [ C_{11}^{-1} + C_{22}^{-1} - C_{12}^{-1} - C_{21}^{-1} ] = \frac{Q^2}{8\pi\epsilon_0} (1/a - 1/b).
\]

(15)

By definition, this is equal to \( Q^2/(2C) \), i.e.

\[
C = \frac{4\pi\epsilon_0 ab}{b - a}.
\]

(16)

Notice that, in this configuration, the potential of the outer sphere is zero and the potential of the inner sphere is \( Q/(4\pi\epsilon_0)(1/a - 1/b) \). Unlike a parallel plate capacitor, the potentials of the two spheres are not equal and opposite.

Zangwill 6.4 The bound charge density is \( \rho_b = -\nabla \cdot P \). Because of the spherical symmetry of the problem, it is easy to apply Gauss’s law: the net flux of polarization out of a spherical annulus between radius \( r \) and \( r + dr \) is \( P[4\pi(r + dr)^2 - 4\pi r^2] = 8\pi Pr^2 dr \). Equating this to \( -\rho_b (4\pi r^2 dr) \), we obtain

\[
\rho_b = -2P/r.
\]

(17)

The surface charge density on the inner surface is \( P \cdot \hat{n} = -P \).

If the hole only extends out to \( r = R_1 \), the surface bound charge density on the outer surface is \( P \). The total bound charge on the outer and inner surfaces is \( 4\pi P R_1^2 \) and \( -4\pi P R^2 \) respectively. The total bound charge in the annulus is

\[
\int_R^{R_1} (-2P/r)4\pi r^2 dr = -8\pi \int_R^{R_1} Pr^2 dr = -4\pi P(R_1^2 - R^2)
\]

(18)

which cancels the charge inside the annulus. But if the polarized object is infinite in size, there is no outer surface and the total bound charge diverges.