Physics 212: Solutions 6

10.3 a) If we had another semi-infinite solenoid attached to the end of the first, with the same current flowing through it, the magnetic field at the meeting point of the two solenoids would be the same as the magnetic field at a point deep in the interior of either solenoid. But the magnetic field at the meeting point is the sum of the magnetic fields from the two solenoids. By symmetry, the component of the field parallel to the axis of the solenoids, which is what contributes to the flux, is twice that from either one of the solenoids by itself. Therefore the flux from one of the semi-infinite solenoids is half the flux in the interior.

b) Near the middle of the solenoid, the field lines are uniform and parallel to the axis of the solenoid, and entirely contained inside it. As one approaches the ends of the solenoid, the lines splay out. This means that i) the magnetic field along the axis is reduced (and is halved by the time we reach the end of the solenoid ii) the magnetic field also has components in the radially outward direction iii) some of the field lines cut through the solenoid and leak outside.

10.11 The current flows entirely in the $-\hat{\theta}$ direction. At an angle $\theta$, the total current $I$ flows through the line of latitude of length $2\pi R \sin \theta$. Therefore the surface current density on the surface of the sphere is $K = -I/(2\pi R \sin \theta) \hat{\theta}$.

By symmetry, the magnetic field is independent of the azimuthal angle $\phi$. Ampere’s law tells us that, in cylindrical coordinates, an azimuthal magnetic field that is equal to $\mu_0 I/(2\pi \rho)$ outside the sphere and zero inside the sphere will work. In spherical coordinates, this means that $B = \mu_0 I/(2\pi R \sin \theta) \Theta(r - R) \hat{\phi}$.

Using the expression for the curl in spherical polar coordinates,

$$\nabla \times B = \frac{1}{r \sin \theta} \partial_{\theta}(\sin \theta B_\phi) \hat{r} - \frac{1}{r} \partial_r (r B_\phi) \hat{\theta}. \quad (1)$$

The first term is zero everywhere, because $\sin \theta B_\phi \propto 1/r$ is independent of $\theta$. The second term is zero outside and inside the sphere, but on the sphere we have

$$j = \frac{1}{\mu_0} \nabla \times B = -\frac{I}{2\pi r \sin \theta} \partial_r (\Theta(r - R)) \hat{\theta} = -\frac{I}{2\pi R \sin \theta} \delta(r - R) \hat{\theta}. \quad (2)$$

This agrees with the expression for the current density we obtained.

10.15 Choose the axis of the sphere to be in the $z$ direction. The current is entirely in the $\hat{\phi}$ direction. If we choose a small line element $R d\theta$ on the surface of the sphere, the charge that flows past it in a time $dt$ is

$$\frac{Q}{4\pi R^2} (R \sin \theta \omega dt) Rd\theta \quad (3)$$

so that

$$K = \frac{Q}{4\pi R} \omega \sin \theta \hat{\phi}. \quad (4)$$
Eqs. (10.43) and (10.44) from the book can be used here. The tangential fields just inside and outside the sphere are

\[ B_\theta(r = R - \delta) = - \sum_l A_l \left( \frac{1}{R} \partial_\theta P_l(\cos \theta) \right) = \sum_l \frac{A_l \sin \theta}{R} P_l'(\cos \theta) \]  

(5)

and

\[ B_\theta(r = R + \delta) = - \sum_l B_l \left( \frac{1}{R} \partial_\theta P_l(\cos \theta) \right) = \sum_l \frac{B_l \sin \theta}{R} P_l'(\cos \theta) \]  

(6)

so that the discontinuity in \( B_\theta \) is

\[ \sum_l \frac{2l + 1}{l} \frac{B_l \sin \theta}{R} P_l'(\cos \theta) = \mu_0 \frac{Q \omega}{4\pi R} \sin \theta \]  

(7)

from the expression for the current. Comparing the two sides, \( 3B_1 = \mu_0 Q \omega / (4\pi) \), and all other coefficients are zero. Therefore inside the sphere, \( \psi = A_1 z/R = -2B_1 z/R \), and outside the sphere \( \psi = B_1 (R^2/r^3) z \). The magnetic field is then

\[ B_{\text{in}} = (2B_1/R) \hat{z} = \frac{\mu_0 Q \omega}{6\pi R} \hat{z} \]

\[ B_{\text{out}} = 3B_1 (R^2/r^3) z \hat{r} - B_1 (R^2/r^3) \hat{z} = \frac{\mu_0 Q \omega R^2}{4\pi r^5} [z \hat{r} - \frac{1}{3} r^2 \hat{z}] \]  

(8)

12.25 Since the density of turns is uniform along the \( z \) axis, equal to \( N/(2R) \), the surface current flowing through a line element \( R \, d\theta \) is \( IN/(2R) R \sin \theta \, d\theta \). Therefore the surface current density is \( NI/(2R) \sin \theta \). This is similar to what we saw in the previous problem, and the magnetic field inside the sphere is therefore

\[ B_{\text{in}} = \frac{\mu_0 NI}{3R} \hat{z} \]  

(9)

The magnetic flux through the turns in a height range \( dz \) is then

\[ \frac{N}{2R} d z B_{\text{in}} \pi R^2 \sin^2 \theta = \frac{NB_{\text{in}}}{2R} \pi R^3 \sin^3 \theta \, d\theta \]  

(10)

which can be integrated to yield a total flux of

\[ \frac{NB_{\text{in}}}{2R} \frac{4\pi R^3}{3} \]  

(11)

The self inductance is then

\[ \frac{2\pi \mu_0 N^2 R}{9} \]  

(12)

14.9 The increase in the magnetic field into the paper means that \( \nabla \times E \) is out of the paper. Therefore an anticlockwise electric field is set up around the loop
containing the two resistors. The total electromotive force around the loop is \(\pi r^2 dB/dt\). This drives current through both the resistors. Therefore the current flowing in the resistors is \(\pi r^2 dB/dt \times 1/(R_1 + R_2)\). The ‘voltage drop’, or more precisely the line integral of the electric field, from the top to the bottom of \(R_1\) is

\[
V_1 = \pi r^2 \frac{dB}{dt} \frac{R_1}{R_1 + R_2}, \tag{13}
\]

while the line integral of the electric field from the bottom to the top of \(R_2\) is

\[
-V_2 = \pi r^2 \frac{dB}{dt} \frac{R_2}{R_1 + R_2}. \tag{14}
\]

14.12 In the presence of magnetic monopoles, Faraday’s law is generalized to (see Eq. (2.68))

\[
\nabla \times E = -\mu_0 J_{mag} - \partial_t B. \tag{15}
\]

Therefore the emf in the loop is equal to \(-\mu_0 I_{mag} - \partial_t \Phi_B\). If the magnetic monopole starts far from the ring and ends up far from the ring, \(\Phi_B \approx 0\) at both times, so that the time integral of the second term on the right hand side is zero. The time integral of the first term is clearly \(-\mu_0 g\). Therefore

\[
\int_{-\infty}^{\infty} E(t) dt = -\mu_0 g = R \int_{-\infty}^{\infty} I(t) dt = QR \tag{16}
\]

where \(Q\) is the total charge that flows past any cross section of the wire loop. Thus \(Q = -\mu_0 g/R\).