15.12 If the current in the outer cylinder flows in the \( z \) direction, \( E = -\lambda/(2\pi \epsilon_0 \rho) \hat{\rho} \) and \( B = -\mu_0 I/(2\pi \rho) \hat{\phi} \) between the cylinders. Therefore the Poynting vector is

\[
S = \frac{1}{\mu_0} E \times B = \frac{\lambda I}{4\pi^2 \epsilon_0 \rho^2} \hat{z}.
\]  

(1)

The power through a cross section between the cylinders is therefore

\[
2\pi \int_{a}^{b} \rho d\rho \frac{\lambda I}{4\pi^2 \epsilon_0 \rho^2} = \frac{\lambda I}{2\pi \epsilon_0} \ln(b/a).
\]  

(2)

The potential difference between the cylinders is

\[
V = \frac{\lambda}{2\pi \epsilon_0} \int_{a}^{b} \frac{d\rho}{\rho} = \frac{\lambda}{2\pi \epsilon_0} \ln(b/a).
\]  

(3)

Therefore the power flowing through a cross section is \( VI \), as it should be.

15.23 To avoid confusion, we change the notation in the problem so that the charge density is \( s \). Therefore by Gauss’ law, the electric field is only non-zero inside the cylinder, where it is equal to \((ps)/(2\epsilon_0)\hat{\rho} \). The electromagnetic momentum density is only non-zero inside the cylinder, where

\[
g = \epsilon_0 E \times B = -\frac{1}{2} (s\rho) B \hat{\phi}.
\]  

(4)

The angular momentum per unit length of the cylinder is

\[
2\pi \int_{a}^{b} \rho d\rho [\rho \times g] = -\pi s B \int_{0}^{a} \rho^3 d\rho \hat{z} = -\frac{1}{4} \pi sa^4 B \hat{z}.
\]  

(5)

When the magnetic field is changed, the torque per unit length is equal to

\[
[(2\pi a^2)\sigma E(a) + 2\pi \int_{0}^{a} \rho^2 s E(\rho) d\rho] \hat{z}.
\]  

(6)

The electric field is \( E(\rho) = -(\rho \partial_t B)/2\hat{\phi} \). Therefore the torque per unit length is

\[
\int_{0}^{a} \rho^3 d\rho \partial_t B = \int_{0}^{a} \rho^3 d\rho \partial_t B = \int_{0}^{a} \rho^3 d\rho \partial_t B = \int_{0}^{a} \rho^3 d\rho \partial_t B = \pi sa^4/4\partial_t B
\]  

(7)

where we have used the fact that \( 2\pi a \sigma = \pi a^2 s \), i.e. \( \sigma = (a/2)s \). Comparing with the electromagnetic angular momentum, we see that the mechanical torque per unit length, i.e. the rate of change of mechanical angular momentum, is equal and opposite to the rate of change of the electromagnetic angular momentum. If the magnetic field is gradually switched off, the mechanical angular momentum
at the end of the process is equal to the electromagnetic angular momentum at
the beginning of the process.

16.15 a) This is obvious by differentiating the expression for \( f(x) \) with respect
to \( x \), which pulls down a factor of \( ik \) inside the integral.

\[
\int f^*(k)g(k)dk = \int \int f^*(k)g(x)e^{-ikx}dkdx = 2\pi \int f^*(x)g(x)dx.
\]  

(8)

c) \[
\langle k \rangle_k = \frac{\int dkf^*(k)kf(k)}{\int dk|f(k)|^2} = \frac{-i\int df^*(x)df/\sqrt{dx}}{\int \sqrt{df(x)|^2}} = -\langle id/dx \rangle_x
\]

(9)

where we have used the results of a) and b). Similarly, we can relate \( \langle k^2 \rangle \) to
\( \int dxh^*(x)h(x) \), and then integrate by parts to relate this to \( \int df^*(x)d^2f(x)/dx^2 \).

\( d \) and \( e \) You will see this in Physics 215 anyway.

4. Let the sphere rotate around the \( z \)-axis. The magnetic field inside a uniformly
magnetized sphere is \( B = 2\mu_0M/3\hat{z} \). The Lorentz force on a charge \( q \) at a
distance \( s \) from the axis of rotation is

\[
q[x\hat{y} - y\hat{x}]\omega \times (B\hat{z}) = qB\omega(x\hat{x} + y\hat{y}).
\]  

(10)

This force must be balanced by the electric force on the charge. Therefore the
electric field at any point inside the sphere is

\[
E(r) = -B\omega(x\hat{x} + y\hat{y}).
\]  

(11)

The charge density inside the sphere is then

\[
\rho = \epsilon_0\nabla \cdot E = -2\epsilon_0B\omega = -\frac{4\mu_0\epsilon_0}{3}M\omega = -\frac{m\omega}{\pi c^2R^3}.
\]  

(12)

The electrostatic potential inside the sphere is obtained by solving the equation
\( -\nabla \varphi = E \). Therefore

\[
\varphi(r) = B\omega(x^2 + y^2)/2 + const = B\omega r^2[1 - \cos^2\theta]/2 + const = -\frac{1}{3}B\omega r^2P_2(\cos \theta) + \frac{1}{3}B\omega r^2 + const.
\]  

(13)

The potential outside the sphere is a solution of Laplace’s equation. Since it
has to match the interior potential on the surface of the sphere, by inspection
we see that it is

\[
(\frac{1}{3}B\omega R^2 + const)\frac{R}{r} - \frac{1}{3}B\omega(R^5/r^3)P_2(\cos \theta).
\]  

(14)

Since the sphere has no net charge, the first (monopole) term in the exterior
potential must be zero, which fixes the constant. Therefore the potential outside
the sphere is

\[
\varphi(r > R) = -\frac{1}{3}B\omega(R^5/r^3)P_2(\cos \theta)
\]  

(15)
and the potential inside the sphere is

$$\varphi(r) = -\frac{1}{3} B \omega r^2 P_2(\cos \theta) + \frac{1}{3} B \omega (r^2 - R^2).$$  \hspace{1cm} (16)$$

The surface charge density is obtained as the discontinuity in the normal component of the radial field, i.e. \( \sigma(\theta) = \epsilon_0 [\partial_r \varphi(r = R^-) - \partial_r \varphi(r = R^+)] \), which is equal to

$$\frac{2}{3} \epsilon_0 B \omega R [1 - P_2(\cos \theta)] - \epsilon_0 B \omega R P_2(\cos \theta) = \frac{2 \epsilon_0 B \omega R}{3} [1 - \frac{5}{2} P_2(\cos \theta)].$$  \hspace{1cm} (17)$$

Using the fact that

$$\frac{2 \epsilon_0 B \omega R}{3} = \frac{2 \epsilon_0 \omega R}{3} \frac{2 \mu_0}{3} M = \frac{\omega m}{3 \pi R^2 c^2}$$  \hspace{1cm} (18)$$

we obtain the expression for \( \sigma(\theta) \) that is sought.

In this case, the electromotive force is obtained from the Lorentz force:

$$E = \int_{\theta=0}^{\pi/2} \omega R \sin \theta B_r(R, \theta) Rd\theta = \omega R^2 B \int_{0}^{\pi/2} \sin \theta \cos \theta d\theta = \frac{\omega BR^2}{2}.$$  \hspace{1cm} (19)$$

This is equal to \( \mu_0 \omega M R^2 / 3 = \mu_0 \omega m / (4 \pi R) \).